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Symmetrical One-Dimensional Cellular Spaces

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It is shown that every one-dimensional cellular space with von Neumann neighborhood (each cell connected to its two immediate neighbors) can be simulated in real-time by a cellular space with a symmetrical local transition function with the same neighborhood, and starting with the same configuration. Symmetrical means that the individual cells cannot distinguish between left and right. © 1985 Academic Press, Inc.

1. INTRODUCTION

Cellular spaces were originally introduced by von Neumann (1966) for modelling self-reproducing systems. Subsequently they have been used as models in various fields, including pattern recognition, array processors, VLSI (systolic algorithms, for example, Forster and Kung (1980)), theoretical biology, and others.

A one-dimensional cellular space can be viewed as an infinite row of (undistinguishable) finite state machines (also called *cells*). These cells are homogenously connected to neighbor cells. At discrete time steps all change their states simultaneously, determining their new ones depending on the states of the neighbor cells directly connected to them. If a cellular space has *von Neumann neighborhood*, each cell is connected to its two immediate neighbors only. Smith (1971) showed that for every one-dimensional cellular space an equivalent cellular space with von Neumann neighborhood can be found. In a symmetrical cellular space the cells cannot tell which of the two input values comes from the right, and which comes from the left neighbor.

Herman (1971, 1972)—mainly motivated by a biological point of view—investigated the abilities of one-dimensional symmetrical spaces by looking at some examples (like simulation of Turing machines, or the well-known French flag and firing squad synchronization problems). In this paper we take a more general approach. Special examples of symmetrical cellular spaces such as solutions to the problems mentioned above, or pattern recognition problems, can then be easily derived using the general method given.

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Investigations of n-dimensional cellular spaces wtih symmetrical, or otherwise restricted transition functions can be found in Szwerinski (1982), where a slightly different approach from the one presented here is used. As is well known, the "Game of Life" (Conway, 1970), is also a special case of a symmetrical 2-dimensional cellular space.

2. Notations and Definitions

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Z}^k$ the k-fold cartesian product of $\mathbb{Z}$. $H_1 := (-1, 0, +1)$ denotes the von Neumann neighborhood index.

Definition 1. Let S be an alphabet. $c_S : \mathbb{Z} \to S$ is called a *configuration*, and $C_S := \{c \mid c: \mathbb{Z} \to S\}$ is the *set of all configurations* c_S. If the context is clear, the subscript S can be omitted.

Definition 2. $A = (S, N, f)$ is called a *one-dimensional cellular space* iff

—S is an alphabet (the *state alphabet*)

—$N \in \mathbb{Z}^k$, $k \in \mathbb{N}$, and no two components of N are equal (*the neighborhood index*); $N = (n_1, n_2, ..., n_k)$.

—f: $S^k \to S$ the *local transition function* which induces the *global transition function* F: $C_S \to C_S$ according to

$$F(c)(i) := f(c(i+n_1), c(i+n_2), ..., c(i+n_k))$$

for all $c \in C_S$, and all $i \in \mathbb{Z}$.

—there exists $s_0 \in S$ (the *quiescent state*) with

$$f(s_0, s_0, ..., s_0) = s_0.$$

Remark. Configurations $c \in C_S$ with $|\{i \in \mathbb{Z} \mid c(i) \neq s_0\}| < \infty$ are called *configurations with limited support*, and the set of these configurations is denoted $\bar{C}_S$ or $\bar{C}$. In this article only configurations with limited support are considered.

Definition 3. A local transition function f: $S^3 \to S$ is called *symmetrical*, iff for all $a, s, b \in S$ holds: $f(a, s, b) = f(b, s, a)$.

Definition 4. Let $A_1 = (S_1, N_1, f_1)$ and $A_2 = (S_2, N_2, f_2)$ be two cellular spaces with global transition functions F_1 and F_2, and let $\bar{C}_1$, $\bar{C}_2$ denote the set of configurations with limited support for these two spaces.

We say A_2 *simulates* A_1 (in real-time), if there are two functions $G\colon \bar{C}_1 \to \bar{C}_2$ and $H\colon \bar{C}_2 \to \bar{C}_1$ (H could be partially defined) with

$$H(F_2^n(G(c_1))) = F_1^n(c_1) \qquad \text{for all } c_1 \in \bar{C}_1 \text{ and } n \in \mathbb{N}.$$

Definition 4 is a formalization of the notion of two cellular spaces being equivalent.

3. A General Theorem for One-dimensional Cellular Spaces

In this section a method is given on how to construct a cellular space with symmetrical local transition function from a given cellular space with von Neumann neighborhood, where the symmetrical one simulates in real-time the original one, and both start with the same initial configuration.

Theorem. *If $A = (S, H_1, f)$ is a one-dimensional cellular space with von Neumann neighborhood, then there exists a cellular space $A' = (S', H_1, f')$ with a symmetrical function f' that simulates A in real-time, where $G\colon \bar{C} \to \bar{C}'$ is the identical mapping; i.e.,*

$$f'(a, z, b) = f'(b, z, a) \qquad \text{for all} \quad a, z, b \in S',$$

and

$$H(F'^n(c)) = F^n(c) \qquad \text{for all} \quad c \in \bar{C} \quad \text{and} \quad n \in \mathbb{N}.$$

Proof. Let $S' := S^4 \cup S$. The quiescent state of A' is (s_0, s_0, s_0, s_0). It is considered to be equal to s_0. The definition of f' is done in two steps: the definition of the first transition (all arguments are elements of S), and later the definition of all following transitions (all arguments are elements of S^4). Let

$$\cdots s_1 s_2 s_3 s_4 \cdots$$

be (part of) the initial configuration, then after the first step the configuration of A' looks like

$$\cdots \begin{pmatrix} s_1 \to s_2 \\ s_0 \leftarrow s_1 \end{pmatrix} \begin{pmatrix} s_2 \to s_3 \\ s_1 \leftarrow s_2 \end{pmatrix} \begin{pmatrix} s_3 \to s_4 \\ s_2 \leftarrow s_3 \end{pmatrix} \begin{pmatrix} s_4 \to s_5 \\ s_3 \leftarrow s_4 \end{pmatrix} \cdots. \tag{1}$$

A state now contains its previous state and the states of the two neighbors of the cell, where $x \to w$ or $w \leftarrow x$ ($x, w \in S$) is read as "the corresponding cell of A is in state x, and its right neighbor is in state w." As

f' is symmetric, it can not distinguish between left and right, and therefore assumes both possibilities. The two ways of writing a state:

$$\begin{pmatrix} x \to w \\ z \leftarrow y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y \to z \\ w \leftarrow x \end{pmatrix} \tag{2}$$

are considered equal. They are called two different *interpretations* of the same state (a more precise verbal interpretation than the one previous follows.) The formal definition of f' for a, s, $b \in S$ is

$$f'(a, s, b) = f'(b, s, a) := \begin{pmatrix} s \to b \\ a \leftarrow s \end{pmatrix} = \begin{pmatrix} s \to a \\ b \leftarrow s \end{pmatrix}.$$

Let f_{mi} be defined as

$$f_{mi}(a, s, b) := f(b, s, a) \qquad \text{for all} \quad a, b, s \in S,$$

and let F_{mi} be the global transition function of the cellular space $A_{mi} = (S, H_1, f_{mi})$.

LEMMA 1. *If we write s_i^n for $F^n(c)(i)$ (the state of the cell i of A after n transition steps) and t_i^n for $F_{mi}^n(c)(i)$ (the state of the corresponding cell of A_{mi} after n transition steps) for an arbitrary initial configuration $c \in \bar{C}$ and $i \in \mathbb{Z}$, $n \in \boldsymbol{N}$, then f' can be defined in such a way that after $n+1$ transition steps the configuration of A' looks like (when choosing the right interpretation for each cell):*

$$\cdots \begin{pmatrix} s_1^n \to s_2^n \\ t_0^n \leftarrow t_1^n \end{pmatrix} \begin{pmatrix} s_2^n \to s_3^n \\ t_1^n \leftarrow t_2^n \end{pmatrix} \begin{pmatrix} s_3^n \to s_4^n \\ t_2^n \leftarrow t_3^n \end{pmatrix} \begin{pmatrix} s_4^n \to s_5^n \\ t_3^n \leftarrow t_4^n \end{pmatrix} \cdots. \tag{3}$$

If we neglect the difference of one transition step between A' and A, we have as a verbal interpretation for a state of A':

$$\begin{pmatrix} x \to w \\ z \leftarrow y \end{pmatrix}$$

"x is the state of the corresponding cell of A, w is the state of the right neighbor of this cell, y is the state of the corresponding cell of A_{mi}, and z is the state of the left neighbor of this cell." Each state has, according to (2), *two different interpretations, but as Lemma* 4 *will show, it is possible to select the right one for each cell, thus being able to determine the configuration A would have after n transition steps after seeing A' after n transition steps.*

Proof of Lemma 1 *(by induction).* It is obviously true for $n = 0$, as (3) becomes (1) in that case.

In order to define the second part of f' (and prove the induction step for Lemma 1), the concept of a *valid assignment* has to be introduced. Let O, P, $Q \in S^4$, and let I, J, K be interpretations with

$$I(O) := \begin{pmatrix} o_x \to o_w \\ o_z \leftarrow o_y \end{pmatrix}, \qquad J(P) := \begin{pmatrix} p_x \to p_w \\ p_z \leftarrow p_y \end{pmatrix}, \qquad K(Q) := \begin{pmatrix} q_x \to q_w \\ q_z \leftarrow q_y \end{pmatrix}$$

$\langle I, O, J, P, K, Q \rangle$ is called a *valid assignment*, iff

$$\begin{aligned} o_w &= p_x, & p_w &= q_x \\ o_y &= p_z, & p_y &= q_z. \end{aligned} \tag{4}$$

Another way of writing a *valid assignment* is

$$\begin{aligned} &o_x \to o_w = p_x \to p_w = q_x \to q_w \\ &o_z \leftarrow o_y = p_z \leftarrow p_y = q_z \leftarrow q_y \end{aligned} \tag{5}$$

which looks like a cell in state P with two neighbors in state O and Q. In order for an assignment to be valid, the overlapping components of the neighboring states have to identical. For example: if P is the state of a cell i of A', O is the state of the cell $i-1$, and Q is the state of the cell $i+1$, then o_w and p_x both contain the state of the cell i of A, and p_w and q_x both contain the state of the cell $i+1$ of A.

Now f' can be defined for O, P, $Q \in S^4$:

$$f'(O, P, Q) = f(Q, P, O) := \begin{pmatrix} f(o_x, p_x, p_w) \to f(p_x, p_w, q_w) \\ f(p_y, p_z, o_z) \leftarrow f(q_y, p_y, p_z) \end{pmatrix} \tag{6}$$

if $\langle I, O, J, P, K, Q \rangle$ is a valid assignment.

It has to be shown that f' is unambiguously defined, i.e., if there is more than one valid assignment, the result should be independent from the valid assignment chosen, and it has to be shown that there is at least one valid assignment, when we start out with an initial configuration $c \in \bar{C}_S$. Before that, a simple fact about valid assignments is stated in the following lemma.

LEMMA 2. *If* $\langle I, Q, J, P, K, Q \rangle$ *is a valid assignment then* $\langle K', Q, J', P, I', O \rangle$ *is also a valid assignment* (*this is called the* dual assignment), *where*

$$I'(O) = \begin{pmatrix} o_y \to o_z \\ o_w \leftarrow o_x \end{pmatrix}, \qquad J'(P) = \begin{pmatrix} p_y \to p_z \\ p_w \leftarrow p_x \end{pmatrix}, \qquad K'(Q) = \begin{pmatrix} q_y \to q_z \\ q_w \leftarrow q_x \end{pmatrix}.$$

$\langle K', Q, J', P, I', O \rangle$ can be written as

$$q_y \rightarrow q_z = p_y \rightarrow p_z = o_y \rightarrow o_z$$
$$q_w \leftarrow q_x = p_w \leftarrow p_x = o_w \leftarrow o_x$$

which makes Lemma 2 obvious, as Eqs. (4) are reproduced.

Assuming for now that f' is defined unambiguously, as will be shown in Lemma 3, we can continue with the proof of the induction step of Lemma 1 (the step from $n+1$ to $n+2$).

According to the induction premise, after $n+1$ transition steps there is an interpretation for the states of a cell i and its two neighbors so that (see (3)) one can choose

$$I(O) = \begin{pmatrix} s_{i-1}^n \rightarrow s_i^n & \\ t_{i-2}^n \leftarrow t_{i-1}^n & \end{pmatrix}, \qquad J(P) = \begin{pmatrix} & s_i^n \rightarrow s_{i+1}^n \\ t_{i-1}^n \leftarrow t_i^n & \end{pmatrix},$$
$$K(Q) = \begin{pmatrix} s_{i+1}^n \rightarrow s_{i+2}^n \\ t_i^n \leftarrow t_{i+1}^n \end{pmatrix}.$$

This is a valid assignment as

$$o_w = s_i^n = p_x, \qquad p_w = s_{i+1}^n = q_x,$$
$$q_z = t_i^n = p_y, \qquad p_z = t_{i-1}^n = o_y.$$

The result of f' for this case follows from (6):

$$f' = \begin{pmatrix} f(s_{i-1}^n, s_i^n, s_{i+1}^n) \rightarrow f(s_i^n, s_{i+1}^n, s_{i+2}^n) \\ f(t_i^n, t_{i-1}^n, t_{i-2}^n) \leftarrow f(t_{i+1}^n, t_i^n, t_{i-1}^n) \end{pmatrix}.$$

The top two values obviously represent $(s_i^{n+1}, s_{i+1}^{n+1})$, and the bottom two values can be written as

$$f_{mi}(t_{i-2}^n, t_{i-1}^n, t_i^n), f_{mi}(t_{i-1}^n, t_i^n, t_{i+1}^n)$$

which is $(t_{i-1}^{n+1}, t_i^{n+1})$. This proves the induction step of Lemma 1. To complete the proof, it will be shown by Lemma 3 that f' is unambiguous.

LEMMA 3. *f' as defined in* (6) *is unambiguous.*

Proof. There are 16 possible potentially valid assignments: I, J, and K can have two different values independently, and the order of O and Q can be reversed independently from the choices for I, J, K.

As Lemma 2 states, for each valid assignment $\langle I, O, J, P, K, Q \rangle$ there is a dual valid assignment $\langle K', Q, J', P, I', O\rangle$. If we evaluate (6) for the dual assignment we get

$$\begin{pmatrix} f(q_y, p_y, p_z) \rightarrow f(p_y, p_z, o_z) \\ f(p_x, p_w, q_w) \leftarrow f(o_x, p_x, p_w) \end{pmatrix}. \tag{7}$$

According to (2), (6) and (7) are two different interpretations of the same state. Thus we can chose one fixed interpretation for P to eliminate the dual assignments, which leaves now 8 potentially different assignments.

Let us assume $\langle I, O, J, P, K, J\rangle$ is a valid assignment, i.e.:

$$\begin{aligned} o_x \rightarrow o_w &= p_x \rightarrow p_w = q_x \rightarrow q_w \\ o_z \leftarrow o_y &= p_z \leftarrow p_y = q_z \leftarrow q_y. \end{aligned} \tag{5}$$

For all the seven other assignments it has to be shown that if they are valid they produce the same result for f'. Because of symmetries we only have to consider four different ones. First, let us consider those assignments where the order of O and Q is the same as in $\langle I, O, J, P, K, Q\rangle$.

Case 1. Besides $\langle I, O, J, P, K, Q\rangle$, $\langle I', O, J, P, K, Q\rangle$ is a valid assignment, i.e., besides (5), it holds

$$\begin{aligned} o_y \rightarrow o_z &= p_x \rightarrow p_w = q_x \rightarrow q_w \\ o_w \leftarrow o_x &= p_z \leftarrow p_y = q_z \leftarrow q_y. \end{aligned} \tag{8}$$

If we combine Eqs. (5) and (8), $o_z = o_w$ and $o_y = o_x$ follow, which means both interpretations of O are identical, and therefore lead to the same value for f'.

For symmetry reasons, it also follows that if $\langle I, O, J, P, K', Q\rangle$ or $\langle I', O, J, P, K', Q\rangle$ are valid (besides $\langle I, O, J, P, K, Q\rangle$), this leads to the same result for f'.

Now we look at assignments with reverse order of O and Q:

Case 2. Besides $\langle I, O, J, P, K, Q\rangle$, $\langle K, Q, J, P, I, O\rangle$ is a valid assignment, which means, besides (5) it holds

$$\begin{aligned} q_x \rightarrow q_w &= p_x \rightarrow p_w = o_x \rightarrow o_w \\ q_z \leftarrow q_y &= p_z \leftarrow p_y = o_z \leftarrow o_y. \end{aligned} \tag{9}$$

This time it follows that $O = Q$ (both neighbors are in the same state), which means the result of f' is the same for both valid assignments.

Case 3. Besides $\langle I, O, J, P, K, Q\rangle$, $\langle K', Q, J, P, I, O\rangle$ is a valid assignment, which means besides (5) it holds

$$\begin{aligned} q_y \to q_z &= p_x \to p_w = o_x \to o_w \\ q_w \leftarrow q_x &= p_z \leftarrow p_y = o_z \leftarrow o_y. \end{aligned} \tag{10}$$

It follows that $(p_x, p_w) = (p_y, p_z)$, and also $O = Q$. From the first observation it follows that $J(P) = J'(P)$, which means the dual assignment of $\langle K', Q, J, P, I, O\rangle$ is $\langle I', O, J, P, K, Q\rangle$ which leads to Case 1, as the dual assignment produces the same result for f'.

The symmetrical case ($\langle K, Q, J, P, I', O\rangle$ valid besides $\langle I, O, J, P, K, Q\rangle$) is also covered by Case 3, which leaves only one case to consider:

Case 4. Besides $\langle I, O, J, P, K, Q\rangle$, $\langle K', Q, J, P, I', O\rangle$ is valid, which means besides (5) it holds

$$\begin{aligned} q_y \to q_z &= p_x \to p_w = o_y \to o_z \\ q_w \leftarrow q_x &= p_z \leftarrow p_y = o_w \leftarrow o_x. \end{aligned} \tag{11}$$

It follows again $(p_x, p_w) = (p_y, p_z)$, which means $J(P) = J'(P)$. Thus the dual assignment of $\langle K', Q, J, P, I', O\rangle$ is $\langle I, O, J, P, K, Q\rangle$, and the value of f' is the same. This completes the proof of Lemma 3 (f' is unambiguously defined) and therefore of Lemma 1.

To summarize the result: we have shown that after $n+1$ transition steps of A' there is an interpretation for the state of each cell so that the configuration looks like

$$\cdots\begin{pmatrix} s_{i-1}^n \to s_i^n & \\ t_{i-2}^n \leftarrow t_{i-1}^n \end{pmatrix}\begin{pmatrix} s_i^n \to s_{i+1}^n \\ t_{i-1}^n \leftarrow t_i^n \end{pmatrix}\begin{pmatrix} s_{i+1}^n \to s_{i+2}^n \\ t_i^n \leftarrow t_{i-1}^n \end{pmatrix}\cdots. \tag{12}$$

LEMMA 4. *There exists a (recursive) function* $E\colon \bar{C}' \to \bar{C}'$ *which selects the right interpretation for each cell of* A' *in order to obtain* (12).

Proof. Let c denote a configuration of A', then $c(i)$ is the state of the cell i which consists of four components:

$$c(i) = (c_1(i), c_2(i), c_3(i), c_4(i))$$

with the two possible interpretations

$$\begin{pmatrix} c_1(i) \to c_2(i) \\ c_4(i) \leftarrow c_3(i) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_3(i) \to c_4(i) \\ c_2(i) \leftarrow c_1(i) \end{pmatrix}.$$

If $c_1(i) = c_3(i)$, and $c_2(i) = c_4(i)$ then these two interpretations are identical, and the "choice" of the right interpretation is trivial. As we only considered initial configurations with finite support, for almost all cells of A' (those in the quiescent state at least) the right interpretation can be selected. From such a fix-point the right interpretation can propagate to all other cells, as the fix-point already defines two of the four components of each neighboring cell, when we define E (with $x, x', z' \in S$) as

$$\begin{aligned} E(c)(i) &:= \begin{pmatrix} c_1(i) \to c_2(i) \\ c_4(i) \leftarrow c_3(i) \end{pmatrix} \\ &\qquad \text{if } c_1(i) = c_3(i) \text{ and } c_2(i) = c_4(i) \\ &\qquad \text{or } E(c)(i-1) = \begin{pmatrix} x \to c_1(i) \\ z \leftarrow c_4(i) \end{pmatrix} \\ &:= \begin{pmatrix} c_3(i) \to c_4(i) \\ c_2(i) \leftarrow c_1(i) \end{pmatrix} \\ &\qquad \text{if } E(c)(i-1) = \begin{pmatrix} x' \to c_3(i) \\ z' \leftarrow c_2(i) \end{pmatrix}. \end{aligned}$$

Starting with a cell i_0 in the quiescent state we can evaluate E for the cell $i_0 + 1$, as according to (12) at least one of the two cases of the definition of E is valid. Then we can determine E for the cell $i_0 + 2$ and so on.

The states of the corresponding cells of A can be received by taking the first components of the result of E. As the first transition step of A' was not a step corresponding to a step of A, the result obtained like this would be one time step behind. In order to make up for it, $H: \bar{C}' \to \bar{C}$ can perform an additional transition step, when it is defined as

$$H(c)(i) := f(\mathrm{Pr}_1(E(c)(i-1)), \mathrm{Pr}_1(E(c)(i)), \mathrm{Pr}_1(E(c)(i+1))),$$

where Pr_1 denotes the projection to the first component of a quadruple. This completes the proof of Theorem 1:

$$H(F'^n(c)(i)) = F^n(c)(i).$$

Remark. In the case of cellular spaces used for pattern recognition, where the patterns are enclosed between boundary cells and where the acceptance of a pattern is indicated by an acceptance cell next to a boundary cell, the theorem can be modified so that no function H is needed for the symmetrical cellular space A' to obtain the same result as A. The boundary cell represents a fix-point so that the acceptance cell can always reflect the right interpretation.

4. Conclusion

Although at first glance one-dimensional cellular spaces whose cells cannot distinguish between left and right (those with symmetrical local transition function) seem to be inferior to cellular spaces without this restriction, the theorem in this article has shown that regarding their computing capabilities this is not the case.

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